Signatures of Turbulence in Atmospheric Laser Propagation

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ABSTRACT

The signatures of turbulence in atmospheric laser propagation are examined, with a particular focus on the effects of non-Kolmogorov turbulence on laser scintillation and phase fluctuations. Non-Kolmogorov properties of the atmospheric index-of-refraction spectrum are outlined, and it is shown that it may be possible to reproduce these features through broadband power-law forcing of the velocity and temperature fields in turbulent flows. Numerical simulations of homogeneous isotropic turbulence subjected to power-law forcing are used to motivate a spectral model for the kinetic energy, which is then extended to address power-law forcing of passive scalars such as the temperature. A modeled non-Kolmogorov index-of-refraction spectrum for power-law forced turbulence is proposed, where the model spectrum consists of standard Kolmogorov and forcing-dominated contributions. This form could reproduce the experimentally observed signatures of atmospheric turbulence on laser propagation and it may provide insights into the origins of non-Kolmogorov turbulence in the atmosphere.

Keywords: Atmospheric Turbulence, Non-Kolmogorov Turbulence, Laser Propagation, Scintillation

1. INTRODUCTION

Laser propagation in the atmosphere is important for a number of different applications, including communications, remote sensing, adaptive optics, and weapon systems.\(^1\) Propagation of lasers through the atmosphere presents special problems however, since atmospheric turbulence can often have undesirable effects on beam properties. These effects include\(^1\) beam spreading and wander, loss of spatial coherence, and small-scale fluctuations in the intensity of the beam, commonly referred to as scintillation. Atmospheric turbulence can also produce fluctuations in the phase of the beam. These various effects are the signatures of turbulence in atmospheric laser propagation. This paper examines in particular how types of non-classical turbulence found in the atmosphere affect scintillation and phase fluctuations.

Atmospheric turbulence affects beam properties primarily through stochastic variations in the optical index of refraction \(n(x, t)\),\(^1,2\) where \(x\) and \(t\) denote space and time coordinates, respectively. These variations are due in large part to fluctuations in the temperature field \(T(x, t)\), which is, in turn, affected by the turbulent velocity field in the atmosphere.\(^1\) Ultimately, characterizing the effects of atmospheric turbulence on laser propagation requires a detailed prescription for \(n(x, t)\) and its spectrum, from which the scintillation and phase fluctuations of the beam can be predicted.

A great deal of research has focused on measuring and characterizing the spectrum of \(n(x, t)\) at various geographic locations and for different levels of the atmosphere. There is no exact analytic form for this spectrum, however, and most prior forms have been formulated using experimental observations and the statistical theory of turbulence developed by Kolmogorov.\(^3\) This theory provides relatively successful predictions for the wavenumber dependence of turbulence spectra at spatial scales far removed from the largest and smallest scales of the flow. The large-scale behaviors of the spectra remain flow-dependent, however, and often require \textit{ad hoc} treatments. Moreover, experimental studies\(^4\) of the index-of-refraction spectrum in the atmosphere have revealed pronounced departures from the Kolmogorov predictions. Such non-Kolmogorov turbulence, as it is commonly termed, also appears in other areas of physics including, for example, flow through porous media\(^5\) and high-speed reacting flows.\(^6\) Some progress has been made in understanding non-Kolmogorov turbulence by studying the effects
of broadband power-law forcing,\textsuperscript{7–9} for which numerical simulations and theoretical analyses become relatively tractable.

In the present study, we consider the effects of power-law forcing on laser propagation, with a view towards understanding and modeling the effects of non-Kolmogorov atmospheric turbulence on laser propagation. In particular, we use numerical simulations and scaling analyses to motivate a model spectrum for $n(x, t)$ in power-law forced turbulence. The model is then used to predict the effects of power-law forcing on the scintillation and phase fluctuations of a laser beam. This study attempts to give insights into the properties of the non-Kolmogorov spectrum for $n(x, t)$ in the atmosphere, and may provide a mechanism for generating artificial atmospheric turbulence experimentally and computationally.

2. NON-KOLMOGOROV TURBULENCE IN THE ATMOSPHERE

Characterizing the effects of atmospheric turbulence on laser propagation requires a detailed description of the index-of-refraction spectrum $\Phi_n(\kappa, z, t)$, where $\kappa \equiv |\kappa|$ is the spatial wavenumber magnitude and $z$ is the location along the propagation path. Using the Rytov approximations,\textsuperscript{1,2} the variances of the log-amplitude ($\sigma_\chi^2$) and phase ($\sigma_\phi^2$) fluctuations of a Gaussian beam wave are given in terms of $\Phi_n(\kappa, z, t)$ by\textsuperscript{1}

$$
\sigma_\phi^2 (r, k, L) = 2\pi^2 k^2 L \int_0^1 \int_{-\infty}^\infty \kappa \Phi_n(\kappa, \eta) \exp \left( -\frac{\Lambda L \kappa^2 \eta^2}{k} \right) \left\{ I_0 (2\Lambda \kappa \eta) \mp \cos \left[ \frac{L \kappa^2}{k} \eta (1 - \Theta \eta) \right] \right\} d\kappa d\eta,
$$

where $k$ is the optical wavenumber, $z = L$ is the current location of the wave originating at $z = 0$, $r$ is the radial distance from the propagation axis, and $I_0$ is the modified Bessel function of the first kind. The parameters $\Lambda$ and $\Theta$ depend on $L$ and properties of the beam at the receiver.\textsuperscript{1} In limiting cases of planar and spherical waves, $\Lambda = 0$, resulting in $r$-independent variances, while $\Theta = 0$ for a plane wave and $\Theta = 1$ for a spherical wave. In Eq. (1) it is assumed that the turbulence is homogeneous and isotropic, and the time dependence has been neglected (an assumption that is often made in studies of optical propagation through the atmosphere\textsuperscript{1}). The following analysis is limited to homogeneous isotropic turbulence, where homogeneity refers to the spatial translational invariance of the turbulence statistics, while isotropy refers to their rotational and reflectional invariance.\textsuperscript{3,9} The scintillation index $\sigma_\tau^2$ is given by $\sigma_\tau^2 = \exp(4\sigma_\chi^2) - 1$, and for small intensity fluctuations it is common to assume $\sigma_\tau^2 \approx \sigma_\chi^2$. This approximation will not be made here, however, and we will continue to work in terms of $\sigma_\chi^2$ with the understanding that $\sigma_\tau^2$ can be readily obtained.

In order to obtain an analytic expression for $\Phi_n(\kappa, z)$ so that the scintillation and phase fluctuations can be calculated from Eq. (1), we first note that in the atmosphere $n(x)$ can be written in terms of the pressure $P(x)$ and temperature $T(x)$ as\textsuperscript{1,10}

$$
n(x) = 1 + 77.6 \times 10^{-6} (1 + 7.52 \times 10^{-3} \lambda^{-2}) \frac{P(x)}{T(x)},
$$

where $\lambda$ (\textmu m) is the optical wavelength, $P(x)$ is given in millibars, $T(x)$ is given in Kelvin, and the coefficients have units designed to make $n(x)$ dimensionless. Since $P(x)$ is expected to be locally homogeneous,\textsuperscript{2} stochastic variations in $n(x)$ are mainly due to fluctuations in $T(x)$.\textsuperscript{1} As a result, much of the focus in determining $\Phi_n(\kappa, z)$ is on understanding the statistical properties of $T(x)$, which can be treated as a passive scalar at most altitudes. In particular, measurements of the temperature power spectrum $\Phi_T(\kappa, z)$, together with the relation in Eq. (2),\textsuperscript{1} give an assumed spectrum for $\Phi_n(\kappa, z)$. Although measurements have shown that $\Phi_n(\kappa, z)$ has a complicated dependence on geographic location, meteorological conditions, and altitude, the most commonly used analytical form of the spectrum is still based on the statistical theory of turbulent flows developed by Kolmogorov.\textsuperscript{1}

2.1 The Kolmogorov Index-of-Refraction Spectrum

The statistical theory developed by Kolmogorov\textsuperscript{3} consists of three primary hypotheses that imply scaling relations for various properties of turbulent flows. First, it is assumed that at scales $r$ sufficiently smaller than the largest scale of the flow, denoted $L_0$, the turbulence is universal and isotropic, regardless of the large-scale flow properties. Second, for $r \ll L_0$, there exists a universal equilibrium range where the flow statistics depend only on the kinematic viscosity $\nu$, the kinetic energy dissipation rate $\epsilon$, and the scale $r$. Finally, for scales $l_0 \ll r \ll L_0$,
where \(l_0\) is the viscous dissipation scale, the turbulence statistics depend only on \(\epsilon\) and \(r\). In this intermediate range of scales, termed the inertial range, energy input at the large scales cascades to smaller scales until it is dissipated as heat by viscous processes at scale \(l_0\). These relatively simple hypotheses give rise to a number of scaling laws for structure functions and spectra, and they have been examined for a wide range of flows.  

The spectrum \(\Phi_\varphi(\kappa, z)\) can be derived by using the theory developed by Kolmogorov, and extended by Obukhov to formulate a relation for the index-of-refraction structure function \(D_n(x, r) = \langle |n(x + r) - n(x)|^2 \rangle\). The structure function \(D_u(x, r) = \langle |u(x + r) - u(x)|^2 \rangle\) of the velocity field \(u(x)\) is given in the inertial range using the Kolmogorov hypotheses as

\[
D_u(r) = C_u^2 r^{2/3}, \quad l_0 \ll r \ll L_0,
\]

where \(r = |r|\), \(C_u^2\) is a coefficient that depends on \(\epsilon\), and the turbulence is assumed to be homogeneous and isotropic. Obukhov obtained a similar expression for the temperature structure function \(D_T(x, r) = \langle |T(x + r) - T(x)|^2 \rangle\) as

\[
D_T(r) = C_T^2 r^{2/3}, \quad l_0 \ll r \ll L_0,
\]

from which the index-of-refraction structure function is assumed to be

\[
D_n(r) = C_n^2 r^{2/3}, \quad l_0 \ll r \ll L_0.
\]

The scaling of the spectra for \(u(x), T(x),\) and \(n(x)\) can also be written using the Kolmogorov hypotheses, giving a \(\kappa^{-11/3}\) scaling in each case. It has been common, however, to write the coefficients in the spectra using the structure function coefficients in Eqs. (3)-(5). Consequently, here we obtain \(\Phi_\varphi(\kappa)\) from \(D_\varphi(r)\) for an arbitrary variable \(\varphi\) using the relation

\[
\Phi_\varphi(\kappa) = \frac{1}{4\pi^2\kappa^2} \int_0^\infty \frac{\sin(\kappa r)}{\kappa r} \frac{d}{dr} \left[ r^2 \frac{dD_\varphi(r)}{dr} \right] dr.
\]

While the \(r\)-dependence of \(D_\varphi(r)\) predicted by the Kolmogorov hypotheses is the same for \(D_u, D_T,\) and \(D_n\), if we assume a more general power-law structure function of the form

\[
D_\varphi(r) = C_\varphi^2 r^{2/3}, \quad l_0 \ll r \ll L_0.
\]

the corresponding \(\Phi_\varphi(\kappa)\) is given from Eq. (6) by

\[
\Phi_\varphi(\kappa) = \left[ \frac{\alpha(\alpha + 1)}{4\pi^2} \right] \int_0^\infty \frac{\eta^{\alpha - 1} \sin(\eta) d\eta}{\eta} C_\varphi^2 \kappa^{-(3+\alpha)}, \quad L_0^{-1} \ll \kappa \ll l_0^{-1}.
\]

The integral in the square brackets can be solved if \(\alpha\) is restricted to \(-1 < \alpha < 1\), giving \(\Phi_\varphi(\kappa)\) as

\[
\Phi_\varphi(\kappa) = \left[ \frac{\alpha(\alpha + 1)}{4\pi^2} \Gamma(\alpha) \sin \left( \frac{\pi \alpha}{2} \right) \right] C_\varphi^2 \kappa^{-(3+\alpha)}, \quad -1 < \alpha < 1, \quad L_0^{-1} \ll \kappa \ll l_0^{-1}.
\]

Since \(\alpha = 2/3\) in the case of Kolmogorov turbulence from Eq. (5), we obtain \(\Phi_\varphi(\kappa, z)\) as

\[
\Phi_\varphi(\kappa, z) = 0.033 C_\varphi^2 (z) \kappa^{-11/3}, \quad L_0^{-1} \ll \kappa \ll l_0^{-1}.
\]

where we have taken into account the possible variation of the coefficient \(C_\varphi^2 (z)\) along the propagation path. The form for \(\Phi_\varphi(\kappa, z)\) in Eq. (10) has been overwhelmingly popular in studies of atmospheric laser propagation, and can be used in Eq. (1) to obtain predictions of the scintillation index and phase properties of laser beams. For instance, \(\sigma_\chi^2\) is written using Eq. (10) for a plane wave as

\[
\sigma_\chi^2(k, L) = 0.563 k^{7/6} \int_0^L C_\varphi^2 (z) (L - z)^{5/6} dz, \quad L_0^{-1} \ll \kappa \ll l_0^{-1},
\]

and Stribleng et al. and Beland have derived extensions of this result for the general power-law scaling given by Eq. (9).
2.2 Non-Kolmogorov Index-of-Refraction Spectrum in the Atmosphere

While the variation of the structure constant $C^n_2(z)$ in Eq. (10) with altitude and atmospheric conditions has received considerable attention (e.g., Chang et al.\textsuperscript{15}), the wavenumber scaling of $\Phi_n(\kappa, z)$ is also important in determining the scintillation and phase fluctuations from Eq. (1). Variations in the spectral balance, which can be caused by $\alpha \neq 2/3$ in Eq. (9), can have a substantial effect on the resulting beam properties, particularly when $\alpha < 2/3$ and the small scales become stronger relative to the large scales.\textsuperscript{1,12,14} Such departures from the classical scaling used to obtain Eq. (10) are manifestations of non-Kolmogorov turbulence, and imply that the assumptions contained in the Kolmogorov hypotheses have in some sense been violated.

Studies have revealed two distinctly non-Kolmogorov characteristics of $\Phi_n(\kappa, z)$ in the atmosphere. First, the wavenumber scaling of $\Phi_n(\kappa, z)$ in the inertial range departs from the Kolmogorov result $\kappa^{-11/3}$ above the atmospheric boundary layer. In particular, there is a $\kappa^{-10/3}$ scaling in the tropopause and a $\kappa^{-5}$ scaling in the stratosphere.\textsuperscript{13} Zilberman, et al.\textsuperscript{13,16} note that the tropopause scaling can be attributed to helical turbulence, while the scaling in the stratosphere is characteristic of anisotropic turbulence (see also Elperin et al.\textsuperscript{17}). In order to address these various scaling regimes, Zilberman et al.\textsuperscript{16} proposed a three-layer index-of-refraction spectrum where the scaling exponent is a function of altitude. Earlier analyses by Stribling et al.\textsuperscript{12} and Beland\textsuperscript{14} adopted a somewhat similar approach and modeled the observed index-of-refraction spectrum using a general spectrum of the form in Eq. (9), from which measures of the scintillation and phase fluctuations in non-Kolmogorov turbulence can be obtained.

The other notable occurrence of non-Kolmogorov turbulence in the atmosphere consists of a departure from simple power-law scaling altogether. Studies in the atmospheric boundary layer,\textsuperscript{4} where the scaling exponent of the index-of-refraction spectrum is nominally the Kolmogorov value $-11/3$, have revealed the presence of a high-wavenumber “bump” in $\Phi_n(\kappa, z).$\textsuperscript{18,19} While Hill\textsuperscript{18} developed a numerical procedure for obtaining this high-wavenumber bump, Andrews\textsuperscript{20} proposed the alternative approximate form

$$\phi_n(\kappa, z) = 0.033 C^n_2(z) \left[ 1 + 1.802 \left( \frac{\kappa}{\kappa_l} \right) - 0.254 \left( \frac{\kappa}{\kappa_l} \right)^{7/6} \exp \left( -\frac{\kappa^2 / \kappa_l^2}{(\kappa^2 + \kappa_0^2)^{11/6}} \right), \quad 0 \leq \kappa < \infty, \quad \kappa_l = 3.3 / \ell_0, \right.$$

which is commonly termed\textsuperscript{4} the modified atmospheric spectrum. The terms in square brackets in Eq. (12) create the high-wavenumber bump since, for example, the $\kappa$ term becomes leading relative to the first term for $\kappa > \kappa_l / 1.802$. From a physical standpoint, the bump represents an increased contribution of the small-scales to $n(x)$, which can have a substantial impact on the resulting scintillation of the laser beam.\textsuperscript{3}

As we will show, it may be possible for spectral features such as non-Kolmogorov scaling exponents and high-wavenumber bumps to be created by broadband power-law forcing of turbulence. While power-law forcing is not necessarily the cause of the non-Kolmogorov features observed in the atmosphere, understanding the effects of this forcing on $\Phi_n(\kappa, z)$ in an idealized setting may yield progress in the study of atmospheric effects on laser propagation. In particular, power-law forcing could provide a convenient mechanism for modeling non-Kolmogorov turbulence in experimental, computational, and theoretical studies of realistic atmospheric effects on laser propagation.

### 3. POWER-LAW FORCED NON-KOLMOGOROV TURBULENCE

The classical picture of energy cascading from large to small scales in turbulence can be disrupted if energy is input directly into the inertial range or over a wide range of scales. Depending on the nature of the energy input, the result can be a breakdown of the Kolmogorov hypotheses. In practical applications, such inertial range or broadband forcing can occur, for example, in flow through an irregular grid,\textsuperscript{5} flow over an urban or vegetative canopy,\textsuperscript{21} and turbulence in the presence of repeated shock-flame interactions.\textsuperscript{6} With respect to atmospheric turbulence, Pouquet et al.\textsuperscript{22} noted the possibility of a spectral gap resulting from turbulence forced at two widely different scales.

Understanding multiscale-forced flows typically requires detailed computational or experimental studies, and it is not always clear how the forcing spectrum resulting from a given physical mechanism can be parameterized. Nonetheless, turbulence subjected to random power-law forcing with a one-dimensional spectrum $E_f(\kappa) \sim \kappa^p$
provides a more tractable basis for studying non-Kolmogorov turbulence. Here \( E(\kappa) \) and its three-dimensional counterpart, \( \Phi(\kappa) \), are related in homogeneous isotropic turbulence by

\[
\Phi(\kappa) = -\frac{1}{2\pi \kappa} \frac{dE(\kappa)}{d\kappa},
\]

and, as a result, the Kolmogorov scaling \( \Phi(\kappa) \sim \kappa^{-11/3} \) corresponds to \( E(\kappa) \sim \kappa^{-5/3} \). The forcing \( E_f(\kappa) \sim \kappa^p \) was first examined analytically by Forster et al.\(^{23}\) and De Dominicis et al.\(^{24}\) (see also Frisch\(^9\)), and has more recently been examined using numerical simulations by Sain et al.\(^7\) and Biferale et al.\(^8,25\). Many of these prior analyses have focused on the somewhat different issue of the relation between power-law forcing and anomalous scaling exponents in turbulent flows. There are, however, indications\(^{25}\) that the forced turbulence spectra may be written in terms of the forcing exponent \( p \). This then introduces the possibility of connecting physical forcing mechanisms with certain aspects of the resulting non-Kolmogorov turbulence.

In the following, we use results from numerical simulations of random power-law forced homogeneous isotropic turbulence to examine the relationship between the forcing exponent \( p \) and the resulting wavenumber scaling of the kinetic energy spectrum \( E(\kappa) \). From analysis of the numerical results, and by drawing upon prior work on power-law forced turbulence, a model for the kinetic energy spectrum in terms of the forcing exponent \( p \) is proposed. This model is based on a superposition of Kolmogorov and forcing-dominated power laws that, while not rigorously derived, does qualitatively reproduce several key features observed in the numerical simulations. This model is extended to the representation of passive scalar spectra in the presence of power-law forcing, which is then finally used as the basis for a non-Kolmogorov form for \( \Phi_n(\kappa, z) \).

### 3.1 Numerical Simulations of Power-Law Forced Turbulence

Numerical simulations of homogeneous isotropic turbulence have been carried out for large-scale forced turbulence as well as randomly-forced turbulence for \( p = -2, -1, -0.5, 0 \). In the simulations, energy is dissipated at the small scales by numerical viscosity, allowing the development of an inertial range without substantially increasing the demand for computational resources. The large-scale forcing is carried out by injecting energy at wavenumber \( \kappa_L \) corresponding to the largest scale of the flow, \( L_0 \). This particular type of forcing is commonly used in numerical simulations of homogeneous isotropic turbulence and gives \( E(\kappa) \sim \kappa^{-5/3} \) in the inertial range, as predicted by the Kolmogorov hypotheses. Power-law forcing is carried out by perturbing the velocity fields with divergence-free Gaussian white noise scaled by the spectrum

\[
E_f(\kappa) = \begin{cases} C_f \kappa^p & \text{for } \kappa_l \leq \kappa \leq \kappa_u, \\ 0 & \text{otherwise} \end{cases}
\]

where \( C_f \) is a coefficient related to the energy input rate of the forcing and \( \kappa_l \) and \( \kappa_u \) denote the lower and upper cutoffs, respectively, of the forcing spectrum. In all of the simulations \( \kappa_l = 2\pi/L_0 \) and \( \kappa_u = \pi/\Delta x \), where \( \Delta x = L_0/N \) is the grid scale and \( N \) is the number of grid points along one side of the computational domain. The computational dimensions for each simulation are \( N^3 = 256^3 \), and the simulations are carried out for five to ten eddy-turnover times. The same energy injection rate is used in all cases, which effectively sets \( C_f \) for each value of \( p \).

Figure 1(a) shows \( E(\kappa) \) for the large-scale and power-law forced simulations. As expected, the Kolmogorov scaling \( E(\kappa) \sim \kappa^{-5/3} \) is obtained for the large-scale case. For power-law forcing with \( p = -2 \) and \(-1 \), Fig. 1(a) shows that the resulting energy spectra are nearly identical to the large-scale forced case outside of the largest scales of the flow. As \( p \) increases, however, \( E(\kappa) \) in the intermediate range of scales becomes increasingly flat, which is indicative of a greater contribution of the small-scales to the overall energy content of the flow. The one-dimensional spectra \( E_{11}(\kappa_1) \) in Fig. 1(b), which are expected to scale in an identical manner to \( E(\kappa) \), show a similar dependence on \( \kappa_1 \). In particular, we obtain \( \kappa_1^{-5/3} \) scaling in the large-scale forced case and an increasingly flat spectrum as \( p \) increases. With respect to the structure of the turbulence, Fig. 2 shows that as \( p \) increases, the large scale vortical structures in the flow are successively broken up and replaced by smaller-scale structures. This is consistent with the increasingly dominant small-scale motions shown in the spectra in Fig. 1 for increasing \( p \).
Figure 1. Energy spectra \( E(\kappa) \) (a) and \( E_{11}(\kappa_1) \) (b) for large-scale and power-law forced turbulence. Dashed red lines show power-law scalings from Eq. (15) given by dimensional matching with the forcing spectrum. Spectra are normalized by the magnitude of \( E(\kappa) \) at \( \kappa = 1 \) (a) and \( E_{11}(\kappa_1) \) at \( \kappa_1 = 1 \) (b) for the large-scale (LS) case.

The results in Fig. 1 indicate that there is a critical value of \( p \), measured here to be \( p_c \approx -1 \), below which the kinetic energy spectra for the power-law forced cases are similar to the large-scale forced spectrum outside of the largest scales of the flow. This critical exponent has been observed previously,\(^7,^8\) and separates forcing-dominated and forcing-independent dynamical regimes.\(^{25,26}\) From a physical standpoint, as the small-scale forcing becomes stronger with increasing \( p \), the natural energy cascade of the turbulent flow becomes overwhelmed and the flow enters the forcing-dominated regime. In this regime, which pertains for \( p > p_c \), Fig. 1 shows that the scaling of \( E(\kappa) \) depends on \( p \). In particular, \( E(\kappa) \) becomes increasingly flat in the intermediate range of scales as \( p \) increases, and the energy content in the high wavenumber dissipation range begins to increase. In the forcing-independent regime for \( p < p_c \), however, the kinetic energy spectra scale according to the \( \kappa^{-5/3} \) Kolmogorov prediction regardless of the value of \( p \).

The break-up of concentrated vortical structures shown in Fig. 2 for increasing \( p \) has been previously observed by Sain et al.\(^7\) This result can be understood by noting that as \( p \) increases, the turbulence increasingly displays the characteristics of the forcing field, which in this case is Gaussian white noise.\(^{26}\) The elongated vortical structures characteristic of turbulence are not present in Gaussian white noise fields, resulting in the relative absence of vortical structures with increasing \( p \) shown in Fig. 2. Several other aspects of the resulting turbulence statistics can also be explained by the increasing dominance of the forcing, such as the increasing Gaussianity of the velocity gradient probability distribution functions (not shown here). It should be noted, however, that even for \( p < p_c \), certain aspects of the turbulence statistics may continue to show a certain degree of Gaussianity. For instance, Fig. 2 shows that the destruction of large-scale coherent vortical structures may not occur abruptly, and the qualitative appearance of the fields undergoes a more gradual change.

### 3.2 Scaling Relations for Power-Law Forced Turbulence

With respect to the scaling relations for the kinetic energy spectra in Fig. 1, Forster et al.\(^{23}\) using renormalization group theory and De Dominicis et al.\(^{24}\) using dimensional arguments (see also Frisch,\(^9\) Biferale et al.,\(^8,^{25}\) and Mazzino et al.\(^{27}\)) showed that the kinetic energy spectrum, denoted \( E'(\kappa) \), due to power-law forcing is given by

\[
E'(\kappa) \sim \kappa^{-1+2p/3}.
\]  

(15)

This result pertains in the forcing-dominated regime for \( p > p_c \), where the scaling of \( E(\kappa) \) is assumed to be completely determined by the forcing. At \( p = -1 \), Eq. (15) gives \( E'(\kappa) \sim \kappa^{-5/3} \), which is identical to the scaling predicted by the Kolmogorov hypotheses. Figure 1 shows that Eq. (15) agrees qualitatively with the observed spectra for \( p > p_c \), although for \( E(\kappa) \) in Fig. 1(a) it appears that the scaling exponents predicted by Eq. (15) are
somewhat more negative than those observed from the numerical simulations. Equation (15) has been verified previously, however, by Sain et al.\textsuperscript{7} for $p_c \leq p \leq 2$. In order to obtain accurate predictions of the scaling exponent for $E(\kappa)$ as $p$ increases, Sain et al.\textsuperscript{7} increased the rate at which energy is dissipated at the small scales when $p$ is large. Similar modifications were not attempted in the present simulations and further analysis, which is not carried out here, is required to determine whether the discrepancies between Eq. (15) and the observed $E(\kappa)$ in Fig. 1 can be explained by insufficient small-scale energy dissipation in the simulations.

Nevertheless, the increase in the scaling exponent of $E(\kappa)$ for $p > p_c$ predicted by Eq. (15) is qualitatively consistent with the results in Fig. 1. Since Fig. 1 further indicates that $E(\kappa) \sim \kappa^{-5/3}$ for all $p < p_c$, we can use Eq. (15) to propose an approximate model (see also Biferale et al.\textsuperscript{25}) for $E(\kappa)$ in the presence of power-law forcing as

$$E(\kappa) \sim \kappa^{-5/3} + \kappa^{-1+2p/3}.$$  \hspace{1cm} (16)

In this proposed model, the non-Kolmogorov dimensional scaling in Eq. (15) due to the forcing is combined with the natural scaling associated with the inertial cascade of the turbulence. While an analysis of shell models of turbulence by Biferale et al.\textsuperscript{25} provides some support for Eq. (16), a more rigorous examination of the validity of Eq. (16) remains an important direction for future research. Here, however, we simply note that Eq. (16) provides a plausible qualitative description of the results in Fig. 1, since Eq. (16) implies a critical forcing exponent $p_c = -1$ separating the forcing-dominated and forcing-independent regimes.\textsuperscript{7} For $p < p_c$ the
The dimensional forcing contribution in Eq. (16) is subleading with respect to the Kolmogorov scaling for sufficiently large $\kappa$, and as a result the Kolmogorov $\kappa^{-5/3}$ spectra is recovered, as shown in Fig. 1. For $p > p_\kappa$, however, the forcing term in Eq. (16) becomes the leading contribution to $E(\kappa)$ for sufficiently large $\kappa$ and the resulting kinetic energy spectrum has a scaling exponent that depends on $p$.

With respect to the index-of-refraction spectrum in the atmosphere, direct power-law forcing of passive scalars is possible in addition to forcing of the velocity fields. In such instances, the forcing exponents for both the scalar and velocity fields may play a role in the resulting scalar spectrum. For an arbitrary passive scalar $\varphi$ forced by a spectrum of the form $E_{f,\varphi}(\kappa) \sim \kappa^q$, dimensional matching with the forcing term\textsuperscript{27} gives

$$E_{\varphi}(\kappa) \sim \kappa^{(3+2q-s)/2},$$  \hspace{1cm} (17)$$

where $s$ is obtained from the energy spectrum of the velocity field, denoted $E(\kappa) \sim \kappa^s$. As with the model for $E(\kappa)$ in Eq. (16), the total scalar spectrum $E_{\varphi}(\kappa)$ is modeled here as the superposition of contributions\textsuperscript{25} from the forcing and the natural behavior of the scalar

$$E_{\varphi}(\kappa) \sim \kappa^\gamma + \kappa^{(-3+2q-s)/2},$$  \hspace{1cm} (18)$$

where the $\kappa^\gamma$ term represents the scaling of $\varphi$ in the absence of passive scalar power-law forcing. If power-law forcing is applied to both the velocity and the scalar fields and both fields are in the forcing-dominated regime, then $s = -1 + 2p/3$ from Eq. (15) and the second term on the right-hand side in Eq. (18) can be written\textsuperscript{27} as $\kappa^{-1-p/3+q}$. The exponent $\gamma$ in the first term on the right in Eq. (18) is left undetermined since the natural scaling of $\varphi$ is expected to also depend on the scaling of $E(\kappa)$, and as a result as $s$ varies $\gamma$ may also vary.

For Kolmogorov velocity fields where the velocity is forced only at large scales and $s = -5/3$, we can write $\gamma = -5/3$ and Eq. (18) is then given by

$$E_{\varphi}(\kappa) \sim \kappa^{-5/3} + \kappa^{q-2/3}.$$  \hspace{1cm} (19)$$

This model accounts for effects due to forcing of the passive scalar by a power-law spectrum, where the scalar is advected by an otherwise Kolmogorov velocity field. This suggests that a non-Kolmogorov scalar field can occur in the presence of a Kolmogorov velocity field, a point that has been made previously with respect to non-Kolmogorov index-of-refraction spectra in the atmosphere\textsuperscript{28}. As with the power-law forced velocity field, Eq. (19) suggests the scalar critical exponent $q_c = -1$. For values of $q > q_c$, the forcing term will be the leading contribution to $E_{\varphi}(\kappa)$ for sufficiently large $\kappa$, while for $q < q_c$, the forcing term is subleading.

While there remains some debate as to the validity of the modeled spectra in Eqs. (16) and (19), particularly in two-dimensional turbulence\textsuperscript{27} these two spectra are taken here simply as plausible models for $E(\kappa)$ and $E_{\varphi}(\kappa)$ in power-law forced turbulent flows. Further research is necessary to rigorously assess Eqs. (16) and (19) for a range of forcing exponents, as well as potentially provide more rigorous derivations for these relations. Nevertheless, Eq. (16) does give a qualitatively accurate description of the results in Fig. 1, and while Eq. (19) deserves more detailed study in the future, it is used in the following as the basis for a modeled $\Phi_n(\kappa, z)$ in the presence of power-law forcing.

4. POWER-LAW FORCED INDEX-OF-REFRACTION SPECTRUM

The power-law forced passive scalar model spectrum in Eq. (19) suggests a corresponding index-of-refraction spectrum in the inertial range as

$$\Phi_n(\kappa, z) = 0.033C^2_n(z)\kappa^{-11/3} + C_f(z)\kappa^{q-8/3}, \quad L_0^{-1} \ll \kappa \ll L_0^{-1},$$  \hspace{1cm} (20)$$

where we have used the relation in Eq. (13) to convert the one-dimensional spectrum into its three-dimensional counterpart, and we have introduced the forcing coefficient $C_f(z)$, which may depend on $q$. The first term on the right-hand side of Eq. (20) corresponds to the Kolmogorov spectrum in Eq. (10), while the second term accounts for the power-law forcing. Here we allow for the possibility of $C^2_n(z)$ and $C_f(z)$ varying independently, and for $q > q_c$ the forcing term in Eq. (20) becomes the leading contribution to $\Phi_n(\kappa, z)$ for $\kappa > \kappa_c$, where

$$\kappa_c = \left[\frac{0.033C^2_n(z)}{C_f(z)}\right]^{1/(q+1)}.$$  \hspace{1cm} (21)$$

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The decoupling of $C_2^2(z)$ and $C_f(z)$ is in contrast to the purely power-law forced cases described in the previous section, where the coefficients of the Kolmogorov and forcing-dominated terms are linked. Equation (20) can thus be considered as a model for the spectral contribution of power-law forcing in addition to the classical Kolmogorov spectrum in Eq. (10), which can arise due to other forcing mechanisms (such as large-scale forcing).

It should be noted that the form in Eq. (20) bears some resemblance to the composite spectra proposed by Gurvich and Belen’kii\(^{29}\) and Greenwood and Tarazano,\(^{30}\) although neither of these prior studies were specifically intended to address effects due to power-law forcing. The first study\(^{29}\) accounts for the different scaling exponents in the troposphere and stratosphere, while the second\(^{30}\) addresses the small-wavenumber behavior of $\Phi_n(\kappa, z)$. Certain aspects of the following analysis of Eq. (20) are also connected to prior studies\(^{12,14}\) of the generalized $\Phi_n(\kappa, z)$ in Eq. (9). Once again, however, these prior studies did not address power-law forcing (where, for instance, the general exponent $\alpha$ in Eqs. (7) and (9) would be replaced by an exponent that depends on the forcing exponent $q$), and did not consider a composite spectrum as in Eq. (20).

Using Eq. (20) it is possible to find the corresponding structure function $D_n(r)$ using the relation between $D_\varphi(r)$ and $\Phi_\varphi(\kappa)$ for an arbitrary variable $\varphi$, namely\(^1\)

$$D_\varphi(r) = 8\pi \int_0^\infty \kappa^2 \Phi_\varphi(\kappa) \left[ 1 - \frac{\sin(\kappa r)}{\kappa r} \right] d\kappa.$$  

(22)

Using the relation for $\Phi_n(\kappa)$ in Eq. (20), the resulting form for $D_n(r)$ is obtained using Eq. (22) as

$$D_n(r, z) = C_2^2(z) r^{2/3} + C_f(z) r^{-1/3-q} \left\{ 8\pi \int_0^\infty \eta^{q-2/3} \left[ 1 - \frac{\sin(\eta)}{\eta} \right] d\eta \right\}, \quad l_0 \ll r \ll L_0.$$  

(23)

If we restrict $q$ so that $-7/3 < q < -1/3$, then we obtain

$$D_n(r, z) = C_2^2(z) r^{2/3} + C_f(z) r^{-1/3-q} \left\{ 8\pi \Gamma \left( q - \frac{2}{3} \right) \sin \left[ \frac{\pi}{2} \left( \frac{2}{3} - q \right) \right] \right\}, \quad l_0 \ll r \ll L_0.$$  

(24)

The restriction on the value of $q$ is imposed in part by the restriction of Eq. (20) to the inertial range. Specifying a large-scale form\(^{14}\) for Eq. (20), which is not addressed here, would permit a wider range of $q$ values.

4.1 Effects of Power-Law Forcing on Laser Propagation

Using the non-Kolmogorov power-law form for $\Phi_n(\kappa, z)$ in Eq. (20), $\sigma_\kappa^2$ and $\sigma_\sigma^2$ can be obtained from Eq. (1) for planar and spherical waves (where $\Lambda = 0$) with constant $C_2^2(z)$ and $C_f(z)$ as

$$\sigma_\kappa^2(\kappa, L) = \frac{0.326 F_\Theta(5/6)L^{11/6}k^{3/6}C_n^2}{\pi^2} \int_0^\infty x^{-11/6} [1 + \cos(x)] dx,$$

$$+ \frac{\pi^2 F_\Theta \left( \frac{2-3q}{6} \right) L^{(8-3q)/6}k^{(3q+10)/6}C_f}{\pi^2} \int_0^\infty x^{(3q-8)/6} [1 + \cos(x)] dx,$$

where $L_0^{-1} \ll \kappa \ll l_0^{-1}$ and $q < 8/3$ is imposed by solution of the integral over $\eta$. The function $F_\Theta(x)$ accounts for the different forms for $\sigma_\kappa^2$ and $\sigma_\sigma^2$ in planar and spherical waves\(^{14}\) and is given by

$$F_\Theta(x) = \left\{ \begin{array}{ll} (x + 1)^{-1} & \text{if } \Theta = 0 \text{ (plane wave)}; \\
\Gamma(x + 1)^2/\Gamma(2x + 2) & \text{if } \Theta = 1 \text{ (spherical wave)}. \end{array} \right.$$  

(25)

(26)

Equation (25) shows that the forcing exponent $q$ affects both the dependence on the optical wavenumber $k$ and propagation length $L$.

Characterizing the magnitudes of $\sigma_\kappa^2$ and $\sigma_\sigma^2$ requires solution of the integrals over the dummy variable $x$ in Eq. (25). Calculating $\sigma_\kappa^2$ is complicated, however, by the low-wavenumber divergence of the spectrum in Eq. (20). Resolution of this issue requires specifying a form for $\Phi_n(\kappa, z)$ in the low-wavenumber limit, which has
been carried out previously but is not the focus of the present study. By contrast, it is readily shown that $\sigma^2$ is given in the inertial range by

$$\sigma^2_{n}(k, L) = 0.563F_\theta(5/6)L^{11/6}k^{5/6}C_n^2 + \pi^2 F_\theta \left( \frac{2 - 3q}{6} \right) \Gamma \left( \frac{3q - 2}{6} \right) \sin \left( \frac{(3q - 8)\pi}{12} \right) L^{(8 - 3q)/6}k^{(3q + 10)/6}C_f,$$  \hfill (27)

where $L_0^{-1} \ll \kappa \ll l_0^{-1}$ and $-10/3 < q < 2/3$ is imposed by the integration. This restriction on $q$ is again imposed in part by the lack of suitable high- and low-wavenumber forms for $\Phi_n(\kappa, z)$ in Eq. (20), and does not necessarily represent any physical limitation of the results.\(^{14}\)

From the standpoint of characterizing and modeling the properties of lasers propagating through the atmosphere, power-law forcing provides a mechanism by which the scintillation may vary from that given by the Kolmogorov hypotheses. If we consider the ratio of the forcing-dominated scintillation term in Eq. (27) to the Kolmogorov term, then for a plane wave ($\Theta = 0$) we obtain

$$\frac{(\sigma^2_{n})_{\text{force}}}{(\sigma^2_{n})_{\text{Kolmogorov}}} \sim \frac{C_f}{C_n^2} \frac{1}{(8 - 3q)} \Gamma \left( \frac{3q - 2}{6} \right) \sin \left( \frac{(3q - 8)\pi}{12} \right) \left( \frac{k}{L} \right)^{(1+q)/2}. \hfill (28)$$

This relation indicates that $q$ has an important effect on the relative weighting of the two scintillation terms, particularly with respect to the dependence on $(k/L)$. If $(k/L) > 1m^{-2}$, as is nearly always the case for most reasonable values of $k$ and $L$ (e.g.\(^{14}\) $k = 2\pi/(0.5\mu m)$ and $L = 100 km$) then the ratio in Eq. (28) increases with increasing $q$. That is, when $q$ is large and there is stronger small-scale forcing, the scintillation due to the forcing term can become significant with respect to the scintillation obtained from the Kolmogorov contribution alone. The dependence of the scintillation on the spectral exponent of $\Phi_n(\kappa, z)$ is well-known,\(^{12,14}\) and Eq. (28) indicates that power-law forcing can result in an additional increase in the scintillation index.

### 4.2 Connection to Atmospheric Turbulence

While any particular non-Kolmogorov index-of-refraction spectrum may be the result of a large number of possible forcings, we can consider what type of power-law forcing might yield the high-wavenumber spectral bump represented in the modified atmospheric spectrum in Eq. (12). Within the inertial range, the modified spectrum in Eq. (12) may be approximated by

$$\Phi_n(\kappa, z) \approx 0.033C_n^2(z)\kappa^{-11/3} + \frac{0.05}{\kappa_l}C_n^2(z)\kappa^{-8/3}, \quad L_0^{-1} \ll \kappa \ll l_0^{-1}, \quad \kappa_l = 3.3/l_0, \hfill (29)$$

where we have simplified the terms in square brackets and retained only the $\kappa^1$ term. Comparing with Eq. (20), the second term in Eq. (29) corresponds to an index-of-refraction forcing spectrum with exponent $q = 0$, which indicates constant forcing at all wavenumbers. Based on Eq. (12) we also obtain an approximate value for the coefficient $C_f$ in Eq. (20) as $C_f \approx 0.05C_n^2/\kappa_l$, from which $\kappa_c$ in Eq. (21) is given as $\kappa_c \approx \kappa_l/2$. It should be noted, however, that in order to obtain this relatively large value of $\kappa_c$, it is required that $0.033C_n^2$ and $C_f$ be substantially different. This situation is not expected to occur in a purely power-law forced case alone, where $0.033C_n^2$ and $C_f$ will be closely linked.

With respect to the non-Kolmogorov scaling exponents in the tropopause and stratosphere, these exponents can occur when the forcing-dominated part of Eq. (20) is leading with respect to the Kolmogorov term. This will in general occur whenever $\kappa > \kappa_c$ for $q > q_c$, or when $\kappa < \kappa_c$ for $q < q_c$. Within the tropopause, the observed $\kappa^{-10/3}$ scaling of $\Phi_n(\kappa, z)$ can be reproduced through power-law forcing with exponent $q = -2/3$. Since $q > q_c$ here, the non-Kolmogorov scaling is the leading contribution to $\Phi_n(\kappa, z)$ for $\kappa > \kappa_c$. For stratospheric turbulence where $\Phi_n(\kappa) \sim \kappa^{-5}$, a similar spectrum can be obtained by forcing with $q = -7/3$, except that here the forcing term is leading for small-wavenumbers $\kappa < \kappa_c$. Once again, however, values of $\kappa_c$ far from the largest or smallest scales of the flow require the coefficients of the two terms in Eq. (20) to be substantially different.

It should be noted that this analysis is not necessarily intended as an explanation for the existence of non-Kolmogorov turbulence in the atmosphere. There are a number of different forcing spectra and physical...
phenomena that could conceivably produce non-Kolmogorov scaling exponents and the bump in the spectrum at high wavenumbers. In particular, it is conceivable that the Kolmogorov energy cascade itself, which is assumed to be a fundamental aspect of all turbulent flows, is replaced by modified cascades for helical (in the tropopause) and anisotropic (in the stratosphere) turbulence. However, the analysis here does show that if the temperature field, and hence index of refraction, is forced by a $k^s$ spectrum, then the resulting index-of-refraction spectrum could be made to display non-Kolmogorov spectral features similar to those found in the atmosphere.

5. CONCLUSIONS

The present study has examined the signatures of turbulence in atmospheric laser propagation, with a particular focus on the effects of non-Kolmogorov turbulence on scintillation and phase fluctuations. These effects are considered through computational and theoretical analyses of power-law forced turbulence, from which a model index-of-refraction spectrum is proposed. The model spectrum consists of power-law contributions from both the classical Kolmogorov scaling and the power-law forcing. Using the model spectrum, the scintillation and phase fluctuations of power-law forced turbulence have been examined. The model spectrum also reveals connections to realistic atmospheric spectra, and in particular we show how non-Kolmogorov scaling exponents and high-wavenumber spectral bumps may be obtained for appropriate choices of the power-law forcing.

The present study of power-law forcing does not necessarily suggest, however, that non-Kolmogorov aspects of the atmospheric index-of-refraction spectrum are the result of power-law forcing. Rather, we have simply shown that using the physically plausible model spectrum in Eq. (20), non-Kolmogorov spectral features similar to those found in the atmosphere can be reproduced. The implications of this, as well as a study of the validity of Eq. (20) for power-law forcing, are topics of future research. Generating an index-of-refraction spectrum similar to that found in the atmosphere using power-law forcing does, however, have potential benefit to computational and experimental studies of atmospheric turbulence effects on laser propagation. In such studies, the index-of-refraction spectrum can be varied by altering the power-law of an external forcing mechanism, allowing beam properties such as scintillation and phase fluctuations to be systematically studied.

Finally, the present study requires additional research in several key areas. First, while Fig. 1 and prior studies provide some support for the model spectra in Eqs. (16) and (19), a detailed computational and theoretical study would allow the validity of these models to be more rigorously assessed. Such a study may also allow relations for $E(\kappa)$ and $E_q(\kappa)$ in power-law forced turbulence to be rigorously obtained in terms of the forcing exponents $p$ and $q$. Similarly, experimental or computational investigation of the model in Eq. (20) would be beneficial, particularly with respect to the relative magnitudes of $C_n^2(\kappa, z)$ and $C_f(z)$ for different types of forcing. In particular, it would be interesting to examine the form for $\Phi_n(\kappa, z)$ resulting from simultaneous large-scale and power-law forcing. The present analysis has also not addressed anisotropic or inhomogeneous effects in the atmosphere. Ultimately, however, the present study does suggest that power-law forced turbulence could play an important role in research on the effects of atmospheric turbulence on laser propagation.

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